

The Petrov type of the BMPV metric

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ABSTRACT: We show that the BMPV metric has Petrov type 22. This means that the BMPV metric is less algebraically special than the five-dimensional Schwarzschild metric, which has Petrov type 22.

KEYWORDS: Classical Theories of Gravity, Black Holes.

Contents

1. Introduction	1
2. Review of the Petrov classification in four dimensions	2
3. The Kerr-Newman metric has Petrov type D	2
4. Review of the Petrov classification in five dimensions	3
5. The BMPV metric has Petrov type 22	4
6. Conclusions and topics for further research	5

1. Introduction

In this article, we calculate the Petrov type of the BMPV metric [1], which is the metric of an extremal, charged, rotating black hole in minimal five-dimensional supergravity [2]¹. It turns out that it has Petrov type 22 . To place this result in context, we remark that the five-dimensional Schwarzschild metric, the five-dimensional Reissner-Nordström metric and the five dimensional Myers-Perry metric [5] all have Petrov type 22 . Therefore, adding electric charge *or* rotation to the five-dimensional black hole does not change its Petrov type. However, adding both charge *and* rotation makes the metric less algebraically special. This result is in stark contrast with the behavior of the Petrov type of the analogous four-dimensional metrics. Adding electric charge or rotation or both to the static four-dimensional black hole does not change its Petrov type: the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman metric all have Petrov type D .

This article is organized as follows. In Section 2, we give a short review of the four-dimensional Petrov classification. In Section 3, we calculate the Petrov type of the Kerr-Newman metric. In Section 4, we review the five-dimensional Petrov classification. In Section 5, we calculate the Petrov type of the BMPV metric. We conclude in Section 6.

¹See ref. [3] for the derivation of all supersymmetric solutions in five-dimensional supergravity and [4] for a discussion of general features of black holes in five dimensions.

2. Review of the Petrov classification in four dimensions

The Petrov classification in four dimensions is well-known. However, the Petrov classification in five dimensions, which we review in Section 4, is less known. Therefore, we give a brief review of the four-dimensional Petrov classification to make the similarity with the five-dimensional Petrov classification clear.

The Petrov classification in four dimensions is most easily discussed using two-component spinors [6]. The Weyl spinor C_{ABCD} is the spinor translation² of the Weyl tensor C_{ijkl}

$$C_{ABCD} = \frac{1}{4} C_{ijkl} \sigma_{AB}^{ij} \sigma_{CD}^{kl} .$$

It is completely symmetric. The Petrov type of a Weyl tensor is given by the factorization properties of the associated Weyl polynomial $W = C_{ABCD} x^A x^B x^C x^D$. This polynomial is homogeneous of degree four in two variables. Therefore, it can always be factorized as

$$C_{ABCD} x^A x^B x^C x^D = (\alpha_A x^A)(\beta_B x^B)(\gamma_C x^C)(\delta_D x^D) . \quad (2.1)$$

In this way, we obtain six different Petrov types, see Figure 1.

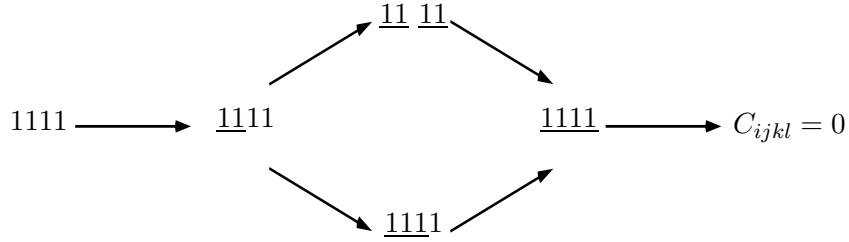


Figure 1: The Penrose diagram of the six different Petrov types in four dimensions. We use underbars to denote how many factors in the factorization (2.1) coincide; the metric has Petrov type $\underline{1111}$, $\underline{1111}$ or $\underline{1111}$ if two, three, or respectively all factors coincide. The metric has Petrov type $\underline{11} \underline{11}$ if two different sets of factors coincide. This case is usually called Petrov type D . Petrov type 1111 is algebraically general, in this case all factors are different.

In the next section, we will also use the Maxwell spinor F_{AB} , which is the spinor translation of the electromagnetic field strength $F_{AB} = \frac{1}{2} \sigma_{AB}^{ij} F_{ij}$. With this bispinor, we associate the Maxwell polynomial $M = F_{AB} x^A x^B$.

3. The Kerr-Newman metric has Petrov type D

We use the following tetrad for the Kerr-Newman metric [7]

$$e^1 = \frac{R}{\rho} (dt - a \sin^2 \theta d\phi), \quad e^2 = \frac{\sin \theta}{\rho} ((r^2 + a^2) d\phi - a dt), \quad e^3 = \frac{\rho}{R} dr, \quad e^4 = \rho d\theta,$$

²We use the following conventions: i, j, k and l are vector indices and A, B, C, D are spinor indices. $\sigma_0 = 1$ and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. The matrices $\bar{\sigma}_i$ are defined by $\bar{\sigma}_0 = \sigma_0$ and $\bar{\sigma}_i = -\sigma_i$ for $i = 1, 2, 3$. Furthermore, we define

$$\sigma_{AB}^{ij} = \epsilon_{AC} \sigma^{iC\dot{C}} \bar{\sigma}_{\dot{C}B}^j .$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $R^2 = r^2 - 2mr + a^2 + Q^2$. This tetrad, together with the gauge potential

$$A = \frac{2Qr}{\rho R} e^1,$$

is a solution of the Einstein-Maxwell equations. A short calculation gives the Maxwell polynomial

$$M = \frac{2iQ}{\bar{z}^2} ((x^1)^2 + (x^2)^2) \quad , \quad \text{where} \quad z = r + ia \cos \theta.$$

It turns out that the Weyl polynomial is proportional to the square of the Maxwell polynomial

$$W = \frac{3\bar{z}(Q^2 - mz)}{2Q^2 z} M^2.$$

The polynomial can be factorized as $\sim (x^1 + ix^2)^2 (x^1 - ix^2)^2$. Therefore, the Kerr-Newman metric has Petrov type D .

4. Review of the Petrov classification in five dimensions

As in four dimensions, the Petrov classification in five dimensions is a classification of the Weyl tensor. We only give a brief review of this classification, a longer discussion can be found in ref. [8]. For a review of the algebraic classification of the Ricci tensor in five dimensions, see ref. [9].

In five dimensions, it is again natural to use spinors to discuss the Petrov classification³. As in four dimensions, we define the Weyl spinor C_{abcd} , which is the spinorial translation of the Weyl tensor C_{ijkl} ,

$$C_{abcd} = (\gamma_{ij})_{ab} (\gamma_{kl})_{cd} C^{ijkl}.$$

Here, $\gamma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$, where γ_i are the γ -matrices in five dimensions⁴. The Weyl spinor is symmetric in all its indices. The Weyl polynomial W is a homogeneous polynomial of degree four in four variables:

$$W = C_{abcd} x^a x^b x^c x^d.$$

As in four dimensions, the Petrov type of a given Weyl tensor is the number and multiplicity of the irreducible factors of its corresponding Weyl polynomial W . In this way, we obtain 12 different Petrov types, which are depicted in Figure 2. We also define the Maxwell spinor $F_{ab} = (\gamma_{ij})_{ab} F^{ij}$ and the Maxwell polynomial $M = F_{ab} x^a x^b$.

³For the Petrov classification in five and higher dimensions using tensor methods: see ref. [10]. It would be good to make a comparison between the five-dimensional Petrov classification using spinors, as discussed in Section 4, and the (more complicated) Petrov classification using tensors.

⁴In Section 5, we will use the following representation $\gamma_1 = i\sigma_1 \otimes 1$, $\gamma_2 = \sigma_2 \otimes 1$, $\gamma_3 = \sigma_3 \otimes \sigma_1$, $\gamma_4 = \sigma_3 \otimes \sigma_2$ and $\gamma_5 = \sigma_3 \otimes \sigma_3$.

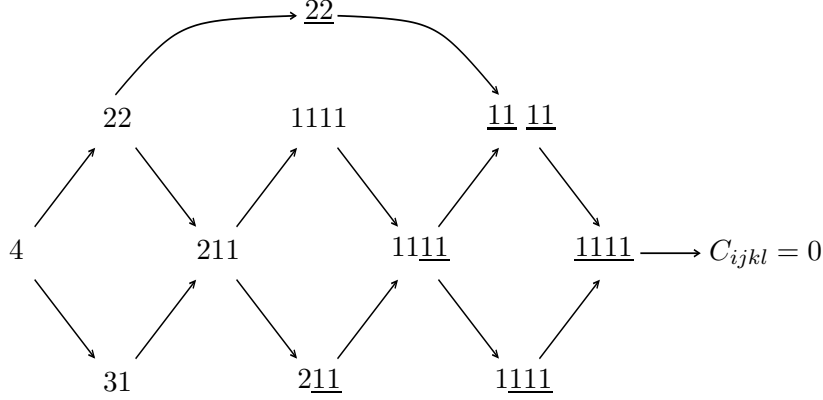


Figure 2: The twelve different Petrov types in five dimensions. We use the following notation. The number denotes the degree of the irreducible factors and underbars denote the multiplicities. For example, a Weyl polynomial which can be factorized into two different factors, each having degree 2, has Petrov type 22. If the two factors of degree 2 are the same, the Petrov type is 22.

5. The BMPV metric has Petrov type 22

We use the following tetrad for the BMPV metric [1]

$$\begin{aligned} e^1 &= f(r)^{-1} \left[dt + \frac{\mu l}{r^2} (\sin^2 \theta d\phi - \cos^2 \theta d\psi) \right], & e^2 &= f(r)^{1/2} r \sin \theta d\phi, \\ e^3 &= f(r)^{1/2} r \cos \theta d\psi, & e^4 &= f(r)^{1/2} dr, & e^5 &= f(r)^{1/2} r d\theta, \end{aligned}$$

with $f(r) = 1 + \frac{\mu}{r^2}$. The gauge potential is $A = \sqrt{3} e^1$. A short calculation gives the Maxwell polynomial

$$\begin{aligned} M &= \frac{4\sqrt{3}\mu}{r^2 f(r)^2} \left[r f(r)^{1/2} ((x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2) \right. \\ &\quad \left. + l \sin \theta ((x^1 + x^2)^2 + (x^3 + x^4)^2) - 2il \cos \theta (x^1 + x^2)(x^3 + x^4) \right]. \end{aligned}$$

The Weyl polynomial is given by $W = MN$, with

$$N = \frac{1}{4}M - \frac{4\sqrt{3}}{r f(r)^{3/2}} [(x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2].$$

Because the Weyl polynomial is the product of two factors of degree two, the Petrov type of the BMPV metric is 22. Hence, we see that the BMPV metric is less algebraically special than the Schwarzschild metric, which has Petrov type 22, see [8]. On the other hand, the five-dimensional Reissner-Nordström metric and the five-dimensional Myers-Perry metric [5] have both Petrov type 22 (the Petrov type of the latter metric has been calculated in ref. [11]). This result is in contrast with the four-dimensional case, where adding charge or rotation or both to the Schwarzschild metric does not change its Petrov type, see Section 3. However, as in four dimensions, it is still true that the Maxwell polynomial divides the Weyl polynomial. A physical reason for this is, as far as I know, not known.

6. Conclusions and topics for further research

In this paper, we have shown that the Petrov type of the BMPV metric is 22. This means that the metric of the charged rotating black hole in five dimensional minimal supergravity is less algebraically special than the five-dimensional Schwarzschild metric. This result is in contrast with the four-dimensional case, where the Schwarzschild metric has the same Petrov type as its charged and rotating cousins. Some topics for further research are the following.

- The BMPV black hole is extremal: its electrical charge is equal to its mass and its two angular momenta are equal. It would be good to calculate the Petrov type of its non-extremal generalizations. These metrics were derived in [12], explicit expressions for some special cases can be found in [13].
- The action of five-dimensional minimal supergravity contains a Chern-Simons term $A \wedge F \wedge F$ with a particular coefficient fixed by supersymmetry. The metric of a charged rotating black hole in five dimensions is not known when this Chern-Simons term has an arbitrary (or even zero) coefficient. One might look for these metrics within the class of metrics of Petrov type 22. As a further simplification, one might even try to assume that the Maxwell polynomial is a factor of the Weyl polynomial – as is the case for the BMPV metric.

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